$(2+1)$ dimensional black hole solutions in $f(R)$ gravity by Noether symmetry

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We obtain general $(2+1)$ dimensional black hole solutions in $f(R)$ theory of gravity by applying the Noether symmetry. Then, we derive generalized anti-de Sitter BTZ type black holes subject to a Noether symmetry. This Noether symmetry is along the vector field ∂_R with a conserved Noether charge associated with the displacement of the cosmological constant. Moreover, we study the thermodynamics of these black holes and show that their entropy can be described by the Cardy-Verlinde formula. This is an evidence for Ads/CFT correspondence in the present $f(R)$ theory of gravity subject to the Noether symmetry.

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I. INTRODUCTION

General relativity in $(2+1)$-dimensional spacetime becomes a topological field theory with only a few nonpropagating degrees of freedom [1]. The vacuum solution of $(2+1)$-dimensional gravity is necessarily flat when the cosmological constant is zero, and it can be shown that no black hole solutions exist [2]. Moreover, the black hole thermodynamics, accounted by quantum states, is ill-defined in this low dimensional model, because of few degrees of freedom. However, it came as a great surprise when $(2+1)$-dimensional BTZ black hole solutions for a negative cosmological constant were shown to exist which can have an arbitrarily high entropy [3]. Indeed, Bañados, Teitelboim and Zanelli [3] have shown that $(2+1)$-dimensional gravity with a negative cosmological constant has a black hole solution, so called BTZ black hole. The BTZ black hole solution in $(2+1)$ dimensional spacetime is derived from a three dimensional action of gravity[1]

$$I = \frac{1}{2}\int dx^3 \sqrt{-g}\,(R-2\Lambda) \tag{1}$$

where $\Lambda = -l^{-2}$ is a negative cosmological constant characterized by a typical length l. The line element in the Schwarzschild coordinates is taken as

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2 \tag{2}$$

where

$$f(r) = \left(-m + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right). \tag{3}$$

This metric is stationary and axially symmetric having just two Killing vectors ∂_t and ∂_ϕ, corresponding to time displacement and rotational symmetry and generically has no other symmetries. Therefore, it is described by two parameters, mass m and angular momentum (spin) J. BTZ black holes are asymptotically anti-de-Sitter (AdS) spacetime with no curvature singularity at the origin, and differ from Schwarzschild and Kerr solutions which are asymptotically flat spacetimes with curvature singularity at the origin. They describe a spacetime of constant negative curvature, with outer and inner horizons, i.e. r_+ (event horizon) and r_- (Cauchy horizon) subject to $J \neq 0$, respectively, given by

$$r_\pm^2 = \frac{l^2}{2}\left(m \pm \sqrt{m^2 - \frac{J^2}{l^2}}\right). \tag{4}$$

The two-parametric family of BTZ black holes, as AdS black holes, play a central role in AdS/CFT conjecture [4] and in brane-world scenarios [5, 6]. The AdS/CFT correspondence has also been generalized for BTZ black holes in higher curvature gravity [7].

Recently, $f(R)$ gravity as a modified theory of gravity has received considerable attention concerning the current acceleration of the universe [8]. On the other hand, Noether symmetry is a physical criterion which allows one to select $f(R)$ gravity models which are compatible with this symmetry [9]. This approach has also been used to obtain $f(T)$ gravity models respecting the Noether symmetry [10]. On the other hand, new spherically symmetric solutions in $f(R)$ gravity have been obtained by Noether Symmetries [11]. In the present paper, we apply the Noether symmetry approach to obtain $(2+1)$ dimensional black hole solutions in $f(R)$ gravity which are consistent with the Noether symmetry.

II. $(2+1)$-DIMENSIONAL $f(R)$ GRAVITY WITH SPHERICAL SYMMETRY

The action in the metric formalism for $(2+1)$ $f(R)$ gravity takes the form

$$I = \frac{1}{2}\int d^3x\sqrt{-g}f(R). \tag{5}$$

This action describes a theory of $(2+1)$ gravity where $f(R)$ is a typical function of the Ricci scalar R. In order

[1] The units $8\hbar G = 1$ has been used.

to study the spherical solutions we take the metric in the following form [3]

$$ds^2 = [-N^2(r) + r^2 M^2(r)]dt^2 + N^{-2}(r)dr^2 \\ +2r^2 M(r)dtd\phi + r^2 d\phi^2, \tag{6}$$

where the radial functions $N(r)$ and $M(r)$ are to be determined as the degrees of freedom. The corresponding Ricci scalar is calculated as

$$R = -\frac{1}{2r}(4rN'^2 + 4rNN'' - r^3 M'^2 + 8NN'), \tag{7}$$

where $'$ denotes the derivative with respect to r. In order to derive the field equations in the $f(R)$ gravity, we generalize the degrees of freedom and define a canonical (point like) Lagrangian $\mathcal{L} = \mathcal{L}(N, M, R, N', M', R')$ so that $\mathcal{Q} = \{N, M, R\}$ is the configuration space and $\mathcal{TQ} = \{N, M, R, N', M', R'\}$ is the related tangent bundle on which $\mathcal{L}$ is defined [9]. Now, we use the method of Lagrange multipliers to set R as a constraint of the dynamics. To this end, by taking a suitable Lagrange multiplier λ and integrating by parts, the Lagrangian becomes canonical and the action takes on the following form

$$\mathcal{S} = \int d^3x\sqrt{-g}[f(R) - \lambda(R + \frac{1}{2r}(4rN'^2 + 4rNN'' \\ -r^3 M'^2 + 8NN'))]. \tag{8}$$

The variation of action with respect to R gives $\lambda = f_R \equiv df/dR$, so the action can be rewritten as

$$\mathcal{S} = \int d^3x\sqrt{-g}[f(R) - f_R(R + \frac{1}{2r}(4rN'^2 + 4rNN'' \\ -r^3 M'^2 + 8NN'))]. \tag{9}$$

Integrating by parts results in the following point-like Lagrangian

$$\mathcal{L} = r(f - Rf_R) + \frac{r^3}{2} f_R M'^2 - 2f_R NN' + 2rf_{RR}R'NN', \tag{10}$$

where $f_{RR} \equiv d^2f/dR^2$. The equations of motion for N, M and R are obtained respectively as

$$N(f_{RRR}R'^2 + f_{RR}R'') = 0, \tag{11}$$

$$(r^3 f_R M')' = 0, \tag{12}$$

$$-rRf_{RR} + \frac{r^3}{2} f_{RR}M'^2 - 4f_{RR}NN' - 2rf_{RR}N'^2 \\ -2rf_{RR}NN'' = 0. \tag{13}$$

III. NOETHER SYMMETRY

Solutions for the dynamics given by the point-like canonical Lagrangian (10) can be obtained by choosing cyclic variables which are related to some Noether symmetries. In general, a non-degenerate point-like canonical Lagrangian $\mathcal{L}$ depends on the variables $q^j(x^\mu)$ and on their derivatives $\partial_\nu q^j(x^\mu)$. Using the Euler-Lagrange equations and after some simple calculations we obtain

$$\partial_\mu\left(\alpha^j \frac{\partial\mathcal{L}}{\partial\partial_\mu q^j}\right) = \alpha^j \frac{\partial\mathcal{L}}{\partial q^j} + \left(\partial_\mu \alpha^j\right) \frac{\partial\mathcal{L}}{\partial\partial_\mu q^j} = L_{\mathbf{X}}\mathcal{L}, \tag{14}$$

where $L_{\mathbf{X}}$ denotes the Lie derivative along the vector field $\mathbf{X}$ defined by

$$\mathbf{X} = \alpha^j \frac{\partial}{\partial q^j} + \left(\partial_\mu \alpha^j\right) \frac{\partial}{\partial\partial_\mu q^j}, \tag{15}$$

which is the generator of symmetry for the dynamics derived by $\mathcal{L}$. This is a statement of *Noether theorem* which asserts that if $L_{\mathbf{X}}\mathcal{L} = 0$, then the Lagrangian $\mathcal{L}$ is invariant along the vector field $\mathbf{X}$. As a consequence, we can define the current

$$j^\mu = \alpha^j \frac{\partial\mathcal{L}}{\partial\partial_\mu q^j}, \tag{16}$$

which is conserved as

$$\partial_\mu j^\mu = 0. \tag{17}$$

The presence of Noether symmetries allows one to reduce the dynamics and find out exact solutions as well as the analytic form of $f(R)$ [9, 11].

IV. $f(R)$ GRAVITY CONSISTENT WITH NOETHER SYMMETRY

Following [9], we define the Noether symmetry in the present model by a vector field X on the tangent space $T\mathcal{Q} = (M, N, R, M', N', R')$ of the configuration space $\mathcal{Q} = (M, N, R)$

$$X = \alpha\frac{\partial}{\partial N} + \beta\frac{\partial}{\partial M} + \gamma\frac{\partial}{\partial R} + \alpha'\frac{\partial}{\partial N'} + \beta'\frac{\partial}{\partial M'} + \gamma'\frac{\partial}{\partial R'}, \tag{18}$$

such that

$$L_X\mathcal{L} = 0. \tag{19}$$

Therefore, a symmetry exists if one finds solutions of the equation $L_{\mathbf{X}}\mathcal{L} = 0$ for the functions α, β and γ, where at least one of them is different from zero. Imposing (19), we obtain the following system of partial differential equations

$$R'N'\left(\alpha f_{RR} + f_{RR}N\frac{\partial\alpha}{\partial N} + \gamma f_{RRR}N + f_{RR}N\frac{\partial\gamma}{\partial R}\right) = 0, \tag{20}$$

$$R'M'\left(2f_{RR}N\frac{\partial\alpha}{\partial M}+r^2f_R\frac{\partial\beta}{\partial R}\right)=0, \tag{21}$$

$$N'M'\left(2f_{RR}N\frac{\partial\gamma}{\partial M}+r^2f_R\frac{\partial\beta}{\partial N}\right)=0, \tag{22}$$

$$N'\left(\alpha f_R+\gamma f_{RR}N+f_R\frac{\partial\alpha}{\partial N}N\right)=0, \tag{23}$$

$$M'^2\left(\frac{1}{2}\gamma f_{RR}+f_R\frac{\partial\beta}{\partial M}\right)=0, \tag{24}$$

$$N'^2\left(f_{RR}N\frac{\partial\gamma}{\partial N}\right)=0, \tag{25}$$

$$R'^2\left(f_{RR}N\frac{\partial\alpha}{\partial R}\right)=0, \tag{26}$$

$$M'\left(f_RN\frac{\partial\alpha}{\partial M}\right)=0, \tag{27}$$

$$R'\left(f_RN\frac{\partial\alpha}{\partial R}\right)=0, \tag{28}$$

which is subject to the following constraint

$$\gamma Rf_{RR}=0. \tag{29}$$

The case $f_{RR}=0$ leads to the common Einstein-Hilbert action which is not of our interest. Moreover, we take $f_{RR}\neq 0, f_R\neq 0$ and also $M'\neq 0, N'\neq 0, R'\neq 0$. So, (29) gives rise to

$$\gamma=0. \tag{30}$$

Moreover, (26) and (27) leads to

$$\alpha=\alpha(N). \tag{31}$$

Using (24) and (30) we obtain

$$\frac{\partial\beta}{\partial M}=0\Longrightarrow\beta=\beta(N,R). \tag{32}$$

On the other hand, Eq.(22) results in

$$\frac{\partial\beta}{\partial N}=0\Longrightarrow\beta=\beta(R). \tag{33}$$

Finally, using (21) and (31) we have

$$\beta=\beta_0=\text{Concst}. \tag{34}$$

Imposing (20) in (30), we obtain the following result

$$\alpha=\frac{A}{N}, \tag{35}$$

which solves Eq.(23), A being a constant. Using the above results in (18), the vector field X becomes

$$X=\frac{A}{N}\frac{\partial}{\partial N}+\beta_0\frac{\partial}{\partial M}-\frac{AN'}{N^2}\frac{\partial}{\partial N'}. \tag{36}$$

The conserved current (16) is written as

$$\begin{aligned} j^r &= \frac{A}{N}\frac{\partial\mathcal{L}}{\partial N'}+\beta_0\frac{\partial\mathcal{L}}{\partial M'} \\ &= -2A(f_R+rf_{RR}R')+\beta_0r^3f_RM', \end{aligned} \tag{37}$$

whose conservation through (17) results in

$$-2A(f_R+rf_{RR}R')+\beta_0r^3f_RM'=C_1, \tag{38}$$

where C_1 is a constant. So, we have

$$-2A(f_R+rR'f_{RR})=C_1-\beta_0C_2, \tag{39}$$

where according to (12)

$$C_2=r^3f_RM'=\text{Const}. \tag{40}$$

The dynamical equation (11) can be written as

$$(R'f_{RR})'=0, \tag{41}$$

which gives rise to

$$f_R=D_1r+D_2, \tag{42}$$

where D_1, D_2 are the constants of integration. Putting (42) into (40) results in

$$M(r)=-\frac{C_2{D_1}^2\ln(D_1r+D_2)}{{D_2}^3}+\frac{C_2{D_1}^2\ln(r)}{{D_2}^3}-\frac{1}{2}\frac{C_2}{D_2r^2}+\frac{C_2D_1}{{D_2}^2r}. \tag{43}$$

In order to find M and R as functions of r, we follow the procedure as is explained bellow. Using the fact that we are looking for the spherical solutions, one may choose the following ansatz

$$R(r)=Kr^n, \tag{44}$$

from which we find

$$r=\left(\frac{R}{K}\right)^{1/n}, \tag{45}$$

where K is a constant. Putting this into (42) leads to

$$f_R=D_1\left(\frac{R}{K}\right)^{1/n}+D_2. \tag{46}$$

Integration with respect to R yields

$$f(R)=D_1R\left(\frac{n}{n+1}\right)\left(\frac{R}{K}\right)^{1/n}+D_2R+D_3, \tag{47}$$

where D_3 is a constant of integration.

V. BLACK HOLE SOLUTIONS

Using (47) and (44) in the equation of motion (13) we obtain

$$N^2(r)=\frac{1}{4D_2^2r^2}[2C_2^2D_1r\ln\frac{r}{D_1r+D_2}(2D_2+3D_1r) -2KD_2^4r^{n+4}-8D_2^4r(Pr-Q)+C_2^2D_2^2], \tag{48}$$

where P and Q are constants of integrations. Now, (43) and (48) determine the spherical solutions for the metric (6) subject to a specific spherically symmetric Ricci scalar (44). To explore the black hole solutions, the metric (6) can be written in the following convenient form

$$ds^2=-N^2(r)dt^2+N^{-2}(r)dr^2+r^2[M^2(r)dt+d\phi]^2. \tag{49}$$

For given constants, D_1, D_2, C_2, P, Q and given values for n, the shift function $N^2(r)$ vanishes and so the horizons exist for those values of r satisfying the following equation

$$2C_2^2D_1r\ln\frac{r}{D_1r+D_2}(2D_2+3D_1r)-2KD_2^4r^{n+4} -8D_2^4r(Pr-Q)+C_2^2D_2^2=0. \tag{50}$$

For the limiting case $D_1, D_3\to 0$ and $D_2\to 1$, namely $f(R)\to R$, we have

$$N^2(r)\to\frac{C_2^2}{4r^2}-\frac{K}{2}r^{n+2}+\frac{2Q}{r}-2P. \tag{51}$$

Assuming $n\to 0, Q\to 0$ and using the identifications $2P=m$, $C_2=J$, and $K=-2l^{-2}$, it turns out that $N^2(r)$ and $M(r)$ approach to the well known solutions in the BTZ black hole [3]

$$N^2(r)\to -m+\frac{r^2}{l^2}+\frac{J^2}{4r^2}, \tag{52}$$

$$M(r)\to-\frac{J}{2r^2}, \tag{53}$$

where m and J, respectively are the mass and angular momentum of the black hole, and l^{-2} accounts for a cosmological constant Λ. The constants of integration m and J are the conserved charges associated with asymptotic invariance under time displacements and rotational invariance, respectively. Note that, in this specific case, $R=-2l^{-2}=2\Lambda=\text{Const}$. Therefore, we find an important result that the BTZ black hole within Eisnstein-Hilbert theory of gravity with a cosmological constant $\Lambda=-l^{-2}$ [3] is the limiting case of the black hole obtained by letting $f(R)\to R=2\Lambda$ in modified $f(R)$ theory of gravity subject to the Noether symmetry along the vector field ∂_R.

The roots of Eq.(51) with $n\to 0, Q\to 0$, are obtained as

$$r_\pm=\sqrt{-\frac{2}{K}}\left[P\left(1\pm\sqrt{1-\left(\frac{C_2}{2P\sqrt{-\frac{2}{K}}}\right)^2}\right)\right]^{\frac{1}{2}}. \tag{54}$$

On the other hand, g_{00} in the metric (6) vanishes at

$$r_{erg}=l\sqrt{m}=2\sqrt{-\frac{P}{K}}, \tag{55}$$

where $r_\pm$ and r_{erg} obey

$$r_-\leq r_+\leq r_{erg}. \tag{56}$$

Similar to the Kerr metric in 3+1 dimensions, r_+ is the black hole horizon, r_{erg} is the surface of infinite redshift and the region between r_+ and r_{erg} is called *ergosphere*. The horizon exists provided

$$P>0,\quad |C_2|<2P\sqrt{-\frac{2}{K}}. \tag{57}$$

Both roots of $N^2=0$ coincide for the extreme case $|C_2|=2P\sqrt{-\frac{2}{K}}$.

Now, let us assume $n\to 0, Q\neq 0$; then we have

$$N^2(r)\to -m+\frac{r^2}{l^2}+\frac{J^2}{4r^2}+\frac{2Q}{r}, \tag{58}$$

$$M(r)\to-\frac{J}{2r^2}. \tag{59}$$

The last term in (58) is novel with respect to the BTZ black hole. This term has a constant Q which may play the role of a *geometric mass*. The presence of this term alters the horizons of BTZ black hole and gives them as four real roots of the following equation

$$4\,r^4-4\,l^2mr^2+8\,l^2Qr+J^2l^2=0. \tag{60}$$

The location of surface of infinite redshift r_{erg}, is also obtained as the solution of the following equation

$$\frac{r^2}{l^2}+\frac{2Q}{r}-m=0. \tag{61}$$

Similar to BTZ black hole, if we let l grow towards very large values, the black hole exterior is pushed away to infinity and we just get the inside. However, to make the black hole disappear as $r_+\to 0$ for the vacuum state, letting $m\to 0$ and $J\to 0$ does not suffice, unless we let $Q\to 0$ as well. *In other words, the characteristics $m\to 0$ and $J\to 0$ of the BTZ black hole resulting in the line element for the vacuum sate*

$$ds^2_{vac}=-\frac{r^2}{l^2}dt^2+\frac{l^2}{r^2}dr^2+r^2d\phi^2, \tag{62}$$

can not provide the vacuum sate in the black hole obtained by letting $f(R)\to R$ in modified $f(R)$ theory of gravity subject to the Noether symmetry. In the latter case, the line element obtained by letting $m\to 0$ and $J\to 0$ is

$$ds^2=-\left(\frac{r^2}{l^2}+\frac{2Q}{r}\right)dt^2+\left(\frac{r^2}{l^2}+\frac{2Q}{r}\right)^{-1}dr^2+r^2d\phi^2, \tag{63}$$

which does not account for the vacuum state $r_+ \to 0$. In fact, this metric still has nonvanishing horizon $r_+ \neq 0$ which is obtained by solving the following equation for $Q < 0$

$$\frac{r^2}{l^2} + \frac{2Q}{r} = 0. \tag{64}$$

To obtain the vacuum state $r_+ \to 0$, we need to let $Q \to 0$ in (63). Therefore, unlike the BTZ black hole in which the zero point of energy has been set so that the mass vanishes when the horizon size goes to zero, here the horizon does not vanish unless other than the mass m, the geometric mass Q vanishes too. Note that the black hole described by the metric (63) with $Q \neq 0$ has a specific feature similar to the Schwarzschild black hole in that the location of surface of infinite redshift r_{erg} coincides with the location of horizon r_+, both of which are obtained by (64).

The important question which arises now is about Q. In fact, similar to m and J, it should be a conserved charge associated with a symmetry. We show that such a symmetry exists under a change in the values of the cosmological constant. To this end, we first use (58) and (59) to insert for $N(r)$, $M(r)$, and $N(r)'$ in (36) and express all partial derivatives with respect to r. Then, we use (45) to replace r by R and $\partial/\partial r$ by $\partial/\partial R$, respectively, so that (36) casts in the following form.

$$X = f(R)\frac{\partial}{\partial R}. \tag{65}$$

This shows that X is a killing vector field along with we have a symmetry under a change in the constant values of Ricci scalar. As is shown above, the constant values of Ricci scalar are the constant values of cosmological constant, namely $R = 2\Lambda = \text{Const}$. Hence, we find the important result that Q is the conserved charge associated with the invariance under the displacement of the cosmological constant, so we denote it as Q_c.

Let us now investigate the issue of singularity and horizon in the context of $(2+1)$ dimensional $f(R)$ gravity by Noether symmetry. In general, the singularity and horizon can be studied respectively by using (49) and (50). For the limiting case $f(R) \to R$, and for given values of m, l with $J \neq 0$ and $Q_c \neq 0$, we expect the appearance of naked singularity which is hidden by horizon, provided $m > 0$. In the special case, we consider $m = -1$ and $Q_c = J = 0$ which coincides with the line element in the BTZ solutions. This case accounts for disappearance of the naked singularity and horizon in the BTZ solutions [3]. Now, we consider $m = -1$, $J = 0$ with $Q_c \neq 0$ in the line element (6) which leads to

$$ds^2 = -\left(1 + \frac{r^2}{l^2} + \frac{2Q_c}{r}\right)dt^2 + \left(1 + \frac{r^2}{l^2} + \frac{2Q_c}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{66}$$

The presence of the term $2Q_c/r$ in the latter case drastically changes the results obtained in the former case [3], in that we still get a naked singularity for $Q_c \neq 0$. However, assuming $Q_c < 0$ a horizon is constructed which makes this naked singularity be hidden.

VI. BLACK HOLE THERMODYNAMICS

In this section, we consider the metric (6) with the functions given by (58) and (59) where $m > 0$.

A. Thermodynamical quantities

The mass, angular momentum and area of the black hole are given respectively by

$$m = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2} + \frac{2Q_c}{r_+}, \tag{67}$$

$$J = 2r_+\sqrt{m - \frac{r_+^2}{l^2} - \frac{2Q_c}{r_+}}, \tag{68}$$

$$A_H = 2\pi r_+. \tag{69}$$

By employing the well-known Bekenstein-Hawking area formula, the entropy of black hole is given by [15]

$$S = 4\pi r_+. \tag{70}$$

We can express the mass m in terms of $S > 0$, $J > 0$, and $Q_c > 0$ as

$$m = \frac{S^2}{l^2} + \frac{J^2}{4S^2} + \frac{2Q_c}{S}. \tag{71}$$

The Hawking temperature, angular velocity and heat capacity of the black hole are given respectively by

$$T_H = \left[\frac{\partial m}{\partial S}\right]_{J,Q_c} = \frac{2S}{l^2} - \frac{J^2}{2S^3} - \frac{2Q_c}{S^2}, \tag{72}$$

$$\Omega = \left[\frac{\partial m}{\partial J}\right]_{S,Q_c} = \frac{J}{2S^2}, \tag{73}$$

$$C = T_H\left[\frac{\partial T_H}{\partial S}\right]^{-1}_{J,Q_c} = T_H\left(\frac{2}{l^2} + \frac{3J^2}{2S^4} + \frac{4Q_c}{S^3}\right)^{-1}. \tag{74}$$

The thermodynamic potential conjugate to Q_c is also obtained as

$$\Phi_c = \left[\frac{\partial m}{\partial Q_c}\right]_{S,J} = \frac{2}{S} = A_H^{-1}. \tag{75}$$

B. Cardy-Verlinde Formula

Verlinde has proposed a generalization of the Cardy formula from $(1+1)$ dimensional conformal field theory (CFT) to $(n+1)$-dimensional one [16]. The Cardy-Verlinde formula is given by

$$S_{CFT} = \frac{2\pi R}{\sqrt{ab}}\sqrt{E_C(2E - E_C)}, \tag{76}$$

where E is the total energy, E_C is the Casimir energy, R is the radius of the system, and a and b are arbitrary positive coefficients independent of R and S. The Casimir energy is defined by the violation of Euler relation

$$E_C = n(E + PV - T_H S - \Phi_c Q_c - \Omega J), \tag{77}$$

where the pressure of the CFT is defined as $P = E/nV$. The total energy is given by the following sum

$$E = E_E + \frac{1}{2}E_C, \tag{78}$$

where E_E is the purely extensive part of the total energy E. Also, the Casimir energy E_C and the purely extensive part of energy E_E expressed in terms of the R and S are given by

$$E_C = \frac{b}{2\pi R}S^{1-1/n} \tag{79}$$

$$E_E = \frac{a}{4\pi R}S^{1+1/n}. \tag{80}$$

Using Witten's work on AdS/CFT correspondence [14], the Cardy-Verlinde formula (76) can be derived by use of the thermodynamics of various black holes with AdS asymptotic, in arbitrary dimension [17].

C. Entropy of the generalized BTZ black hole by Cardy-Verlinde formula

The entropy of generalized BTZ black hole with AdS asymptotic described by (58) and (59) can be derived by the Cardy-Verlinde formula (76). We obtain the Casimir energy E_C using (77) where $n = 1$. In so doing, we evaluate the following terms

$$T_H S = \frac{2S^2}{l^2} - \frac{J^2}{2S^2} - \frac{2Q_c}{S}, \tag{81}$$

$$\Phi_c Q_c = \frac{2Q_c}{S}, \tag{82}$$

$$\Omega J = \frac{J^2}{2S^2}. \tag{83}$$

Since the generalized black hole is asymptotically anti-de-Sitter, the total energy is $E = m$ and the Casimir energy is obtained

$$E_C = 2\left(\frac{J^2}{4S^2} + \frac{2Q_c}{S}\right). \tag{84}$$

On the other hand, putting $n = 1$ in (79) leads to

$$E_C = \frac{b}{2\pi R}. \tag{85}$$

By equating the right hand sides of (84) and (85), the radius R is obtained as

$$R = \frac{b}{4\pi}\left(\frac{J^2}{4S^2} + \frac{2Q_c}{S}\right)^{-1}. \tag{86}$$

Moreover, by using $PV = E$, (81), (82), and (83) in (77), the quantity $(2E - E_C)$ is evaluated as

$$2E - E_C = \frac{2S^2}{l^2}. \tag{87}$$

The purely extensive part of the total energy E_E is then obtained by substitution of (87) in (78)

$$E_E = \frac{S^2}{l^2}. \tag{88}$$

On the other hand, putting $n = 1$ in (80) gives

$$E_E = \frac{a}{4\pi R}S^2. \tag{89}$$

By equating the right hand sides of (88) and (89), the radius R is obtained again as

$$R = \frac{a}{4\pi}l^2. \tag{90}$$

By using (86) and (90), the radius expressed in terms of the arbitrary positive coefficients a and b is obtained

$$R = l\frac{\sqrt{ab}}{4\pi}\left(\frac{J^2}{4S^2} + \frac{2Q_c}{S}\right)^{-1/2}. \tag{91}$$

Substitution of (84), (87) and (91) in the Cardy-Verlinde formula (76) gives the following result

$$S_{CFT} = S, \tag{92}$$

which asserts that the entropy of the generalized BTZ black hole can be expressed in the form of Cardy-Verlinde formula.

VII. CONCLUSIONS

In the present paper, we have obtained general $(2+1)$ dimensional black hole solutions in $f(R)$ gravity by using the Noether symmetry. We have also obtained the

important result that the anti-de Sitter BTZ black hole within Eisntein-Hilbert theory of gravity with a negative cosmological constant $\Lambda = -l^{-2}$ is the limiting case of the black hole obtained by letting $f(R) \to R = 2\Lambda$ in modified $f(R)$ theory of gravity subject to the Noether symmetry along the vector field ∂_R, with zero conserved Noether charge. In the nonvanishing case of conserved Noether charge, the anti-de Sitter BTZ black hole solution is generalized to include a new term playing the role of a geometric mass. The presence of this new term alters the horizons of BTZ black hole and its thermodynamics, accordingly. It turns out the the conserved Noether charge is associated with the invariance under the displacement of the cosmological constant.

The present study may also have important impact on the AdS/CFT correspondence from the black hole and its thermodynamics point of view. Witten has argued that the thermodynamics of a certain conformal field theory can be identified with the thermodynamics of black holes in anti-de Sitter space [14]. Here, we have obtained a new class of anti-de Sitter BTZ black holes in modified $f(R)$ theory of gravity subject to the Noether symmetry along the vector field ∂_R which has a conserved Noether charge playing the role of a *geometric mass*. Hence, one may think that the thermodynamics of a certain conformal field theory can be identified with the thermodynamics of a black hole solution obtained in the modified $f(R)$ theory of gravity consistent with such a Noether symmetry. This is an evidence for the validity of Ads/CFT correspondence in the $f(R)$ theory of gravity subject to the Noether symmetry. The rigorous study of such Ads/CFT correspondence is very appealing and needs another work.

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